

A unifying approach to branching processes in varying environment

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Abstract

Branching processes $(Z_n)_{n \geq 0}$ in varying environment generalize the Galton-Watson process, in that they allow time-dependence of the offspring distribution. Our main results concern general criteria for a.s. extinction, square-integrability of the martingale $(Z_n/\mathbf{E}[Z_n])_{n \geq 0}$, properties of the martingale limit W and a Yaglom type result stating convergence to an exponential limit distribution of the suitably normalized population size Z_n , conditioned on the event $Z_n > 0$. The theorems generalize/unify diverse results from the literature and lead to a classification of the processes.

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1 Introduction and main results

Branching processes $(Z_n)_{n \geq 0}$ in varying environment generalize the classical Galton-Watson process, in that they allow time-dependence of the offspring distribution. They constitute an important model for applications, provided that one succeeds in capturing their typical properties in sufficiently large generality. In this paper we strive to furnish such a basis, ranging from criteria for a.s. extinction to Yaglom type results. We require only a mild regularity assumption, in particular we don't set any restrictions on the sequence of expectations $\mathbf{E}[Z_n]$, $n \geq 0$, thereby unifying diverse results from the literature.

In order to define a branching process in varying environment (BPVE) denote by Y_1, Y_2, \dots a sequence of random variables with values in \mathbb{N}_0 and by f_1, f_2, \dots their distributions. Let Y_{in} , $i, n \in \mathbb{N}$, be independent random variables such that Y_{in} and Y_n coincide in distribution for all $i, n \geq 1$. Define the random variables Z_n , $n \geq 0$, with values in \mathbb{N}_0 recursively as

$$Z_0 := 1, \quad Z_n := \sum_{i=1}^{Z_{n-1}} Y_{in}, \quad n \geq 1.$$

Then the process $(Z_n)_{n \geq 0}$ is called a *branching process in varying environment* $v = (f_1, f_2, \dots)$ with initial value $Z_0 = 1$. It may be considered as a model for the development of the size of a

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population where individuals reproduce independently with offspring distributions f_n potentially changing among generations. Without further mention we always require that $0 < \mathbf{E}[Y_n] < \infty$ for all $n \geq 1$.

There is one completely general result on a BPVE due to Lindvall [12]. Building on results on Church [3] it actually requires no assumption at all. It says that Z_n is a.s. convergent to a random variable Z_∞ with values in $\mathbb{N}_0 \cup \{\infty\}$. It also clarifies under which conditions $(Z_n)_{n \geq 0}$ may ‘fall asleep’ at a positive state meaning that the event $0 < Z_\infty < \infty$ occurs with positive probability. Let us call such a branching process *sticky*. Thus for a BPVE it is no longer true that the process a.s. either gets extinct or else converges to infinity. For the readers’ convenience we add as an appendix a (comparatively) short proof of Lindvall’s theorem.

A BPVE may exhibit extraordinary properties, which don’t occur for Galton-Watson processes. Thus it may possess different growth rates, as detected by MacPhee and Schuh [14]. Here we like to establish a framework which leaves such exceptional phenomena out and elucidates the generic behaviour. We accomplish this in an L^2 -setting, and our results show that such a setting can hardly be avoided.

Our main assumption is a requirement of uniformity which reads as follows:

$$\exists c_1 < \infty \forall n \geq 1 : \mathbf{E}[Y_n(Y_n - 1)] \leq c_1 \mathbf{E}[Y_n] \cdot \mathbf{E}[Y_n - 1 \mid Y_n \geq 1] . \quad (\text{A})$$

As we shall explicate in the next section, this regularity assumption is considerably mild. It is fulfilled for distributions f_n , $n \geq 1$, belonging to any common class of probability measures, like Poisson, binomial, hypergeometric, geometric, linear fractional, negative binomial distributions, without any restriction to the parameters. It is also satisfied in the case that the random variables Y_n , $n \geq 1$, are a.s. uniformly bounded by a constant, thereby being advantageous compared with a condition like $\mathbf{E}[Y_n(Y_n - 1)] \leq c \mathbf{E}[Y_n]^2$.

A direct verification of (A) can be tedious in individual cases because of the involved conditional expectation. The following regularity requirement is much more accessible in this respect and easily confirmed for all above examples:

$$\exists c_2 < \infty \forall n \geq 1 : \mathbf{E}[Y_n(Y_n - 1)(Y_n - 2)] \leq c_2 \mathbf{E}[Y_n(Y_n - 1)] \cdot (1 + \mathbf{E}[Y_n]) < \infty . \quad (\text{A}^*)$$

We shall show in the next section that indeed this Assumption (A*) implies (A). We shall also use (A*) in Theorem 4 below.

Before presenting our results let us agree on the following notational conventions: Let \mathcal{P} be the set of all probability measures on \mathbb{N}_0 . We write the weights of $f \in \mathcal{P}$ as $f[k]$, $k \in \mathbb{N}_0$. Also we define

$$f(s) := \sum_{k=0}^{\infty} s^k f[k] , \quad 0 \leq s \leq 1 .$$

Thus we denote the probability measure f and its generation functions by one and the same symbol. This facilitates presentation and will cause no confusion whatsoever. Keep in mind that each operation applied to these measures has to be understood as an operation applied to their generating functions. Thus $f_1 f_2$ stands not only for the multiplication of the generating functions f_1, f_2 but also for the convolution of the respective measures. Also $f_1 \circ f_2$ expresses the composition of generating functions as well as the resulting probability measure. We shall consider the mean and factorial moments of a random variable Y with distribution f ,

$$\mathbf{E}[Y] = f'(1) , \quad \mathbf{E}[Y(Y - 1)] = f''(1) , \quad \mathbf{E}[Y(Y - 1)(Y - 2)] = f'''(1) ,$$

and its normalized second factorial moment and normalized variance

$$\nu := \frac{\mathbf{E}[Y(Y-1)]}{\mathbf{E}[Y]^2}, \quad \rho := \frac{\mathbf{Var}[Y]}{\mathbf{E}[Y]^2} = \nu + \frac{1}{\mathbf{E}[Y]} - 1.$$

We shall discuss branching processes in varying environment along the lines of Galton-Watson processes. Let for $n \geq 1$

$$q := \mathbf{P}(Z_\infty = 0), \quad \mu_n := f'_1(1) \cdots f'_n(1), \quad \nu_n := \frac{f''_n(1)}{f'_n(1)^2}, \quad \rho_n := \nu_n + \frac{1}{f'_n(1)} - 1$$

and also $\mu_0 := 1$. Thus q is the probability of extinction and, as is well-known, $\mu_n = \mathbf{E}[Z_n]$, $n \geq 0$. Note that for the standardized factorial moments ν_n we have $\nu_n < \infty$ under assumption (A). This implies $\mathbf{E}[Z_n^2] < \infty$ for all $n \geq 0$ (see Section 4 below).

Assumption (A) is a mild requirement with substantial consequences, as seen from the following differing necessary and sufficient criteria for a.s. extinction.

Theorem 1. *Assume (A). Then the conditions*

- (i) $q = 1$,
- (ii) $\mathbf{E}[Z_n]^2 = o(\mathbf{E}[Z_n^2])$ as $n \rightarrow \infty$,
- (iii) $\sum_{k=1}^{\infty} \frac{\rho_k}{\mu_{k-1}} = \infty$,
- (iv) $\mu_n \rightarrow 0$ and/or $\sum_{k=1}^{\infty} \frac{\nu_k}{\mu_{k-1}} = \infty$

are equivalent. Moreover, the conditions

- (v) $q < 1$,
- (vi) $\mathbf{E}[Z_n^2] = O(\mathbf{E}[Z_n]^2)$ as $n \rightarrow \infty$,
- (vii) $\sum_{k=1}^{\infty} \frac{\rho_k}{\mu_{k-1}} < \infty$,
- (viii) $\exists 0 < r \leq \infty : \mu_n \rightarrow r$ and $\sum_{k=1}^{\infty} \frac{\nu_k}{\mu_{k-1}} < \infty$

are equivalent.

These conditions are effective in different ways. Condition (iii)/(vii) appears to be particularly suitable as a criterion for a.s. extinction, whereas the conditions (iv) and (viii) will prove useful for the classification of a BPVE. Condition (vi) will allow to determine the growth rate of Z_n , see Theorem 2. Observe that (ii) can be rewritten as $\mathbf{E}[Z_n] = o(\mathbf{Var}[Z_n])$. In simple phrase this says that under (A) we have a.s. extinction, iff the noise dominates the mean in the long run.

In his publication [13] Lyons already obtained the equivalence of (v), (vi), (vii) and (somewhat disguised) (viii) under the assumption that the Y_n are a.s. bounded by a constant. We point out that conditions (iii) and (vi) employ not only the expectations μ_n but also second moments.

This is a novel aspect in comparison with Galton-Watson processes and also with Agresti's [1] classical criterion on branching processes in varying environment. Agresti proves a.s. extinction iff $\sum_{k \geq 1} 1/\mu_{k-1} = \infty$. He may do so by virtue of his stronger assumptions, which e.g. don't cover sticky processes. In our setting it may happen that $\sum_{k \geq 1} \rho_k/\mu_{k-1} = \infty$ and $\sum_{k \geq 1} 1/\mu_{k-1} < \infty$, and also the other way round. This is shown by the following examples.

Examples.

(i) Let Y_n take only the values $n+2$ or 0 with $\mathbf{P}(Y_n = n+2) = n^{-1}$. Then $\mathbf{E}[Y_n(Y_n - 1)] \sim n$, $\mathbf{E}[Y_n] = 1 + 2/n$, $\mathbf{E}[Y_n - 1 \mid Y_n \geq 1] \sim n$ such that (A) is fulfilled. Also $\mu_n \sim n^2/2$ and $\rho_n \sim n$, hence $\sum_{k \geq 1} 1/\mu_{k-1} < \infty$ and $\sum_{k \geq 1} \rho_k/\mu_{k-1} = \infty$.

(ii) Let Y_n take only the values 0, 1 or 2 with $\mathbf{P}(Y_n = 0) = \mathbf{P}(Y_n = 2) = 1/(2n^2)$. Then $\mathbf{E}[Y_n(Y_n - 1)] \sim n^{-2}$, $\mathbf{E}[Y_n] = 1$ and $\mathbf{E}[Y_n - 1 \mid Y_n \geq 1] \sim 1/(2n^2)$ such that (A) is fulfilled. Also $\mu_n = 1$ and $\rho_n \sim n^{-2}$, hence $\sum_{k \geq 1} 1/\mu_{k-1} = \infty$ and $\sum_{k \geq 1} \rho_k/\mu_{k-1} < \infty$. \square

The last example exhibits a sticky branching processes.

Next we turn to the normalized population sizes

$$W_n := \frac{Z_n}{\mu_n}, \quad n \geq 0.$$

Clearly $(W_n)_{n \geq 0}$ constitutes a non-negative martingale, thus there exists an integrable random variable $W \geq 0$ such that

$$W_n \rightarrow W \text{ a.s., as } n \rightarrow \infty.$$

Under (A) the random variable W exhibits the dichotomy known for Galton-Watson processes.

Theorem 2. *Assume (A). Then we have:*

- (i) *If $q = 1$, then $W = 0$ a.s.*
- (ii) *If $q < 1$, then $\mathbf{E}[W] = 1$ and $\mathbf{P}(W = 0) = q$.*

A formula for the variance of W may be found in [7]. A comparison with Lindvall's theorem shows that under (A) the sticky processes are just those processes which fulfil the properties $q < 1$ and $0 < \lim_{n \rightarrow \infty} \mu_n < \infty$. We point out that Assumption (A) excludes the possibility of $\mathbf{P}(W = 0) > q$, as determined in [14] for a special case and further discussed in [5], [4].

Now we consider the random variables Z_n conditioned on the events $Z_n > 0$. The next result specifies the circumstances under which the random variables stay stochastically bounded.

Theorem 3. *Let (A) be satisfied. Then these conditions are equivalent:*

- (i) *for all $\varepsilon > 0$ there is a $c < \infty$ such that $\mathbf{P}(Z_n > c \mid Z_n > 0) \leq \varepsilon$ for all $n \geq 0$,*
- (ii) *there is a $c > 0$ such that $c\mu_n \leq \mathbf{P}(Z_n > 0) \leq \mu_n$ for all $n \geq 0$, or, what amounts to the same thing, $\sup_{n \geq 0} \mathbf{E}[Z_n \mid Z_n > 0] < \infty$,*
- (iii) $\sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}} = O\left(\frac{1}{\mu_n}\right)$ as $n \rightarrow \infty$

This theorem applies to two different regimes. In case of $q < 1$ its conditions are fulfilled if we have $0 < \lim_{n \rightarrow \infty} \mu_n < \infty$, that is if we deal with a sticky process. The case $q = 1$ is more substantial. For a Galton-Watson process the theorem's conditions are valid just in the subcritical setting. Recall that in this special situation the conditioned random variables Z_n have a limiting distribution, too. It is easy to see that such a result cannot hold in our general context of a BPVE. Indeed: there are two offspring distributions \hat{f} and \tilde{f} such that the limiting distributions \hat{g} and \tilde{g} for the corresponding conditional Galton-Watson processes differ from each other. Choose an increasing sequence $0 = n_0 < n_1 < n_2 < \dots$ of natural numbers and consider the BPVE $(Z_n)_{n \geq 0}$ in varying environment $v = (f_1, f_2, \dots)$, where $f_n = \hat{f}$ for $n_{2k} < n \leq n_{2k+1}$, $k \in \mathbb{N}_0$, and $f_n = \tilde{f}$ else. Then it is obvious that $Z_{n_{2k+1}}$ given the event $Z_{n_{2k+1}} > 0$ converges in distribution to \hat{g} and $Z_{n_{2k}}$ given the event $Z_{n_{2k}} > 0$ converges in distribution to \tilde{g} , if only the sequence $(n_k)_{k \geq 0}$ is increasing sufficiently fast.

Finally we arrive at results in the spirit of Kolomgorov's and Yaglom's classical asymptotics, which for Galton-Watson processes signify the critical region. Here we rest on Assumption (A*).

Theorem 4. *Let (A*) be satisfied and let $q = 1$. Assume that*

$$\frac{1}{\mu_n} = o\left(\sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}}\right)$$

as $n \rightarrow \infty$. Then

$$\mathbf{P}(Z_n > 0) \sim 2\left(\sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}}\right)^{-1}$$

as $n \rightarrow \infty$. Moreover, setting

$$a_n := \frac{\mu_n}{2} \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}}, \quad n \geq 1,$$

then $a_n \rightarrow \infty$ and the distribution of Z_n/a_n conditioned on the event $Z_n > 0$ converges to a standard exponential distribution.

Observe that under the assumptions of this theorem $a_n \sim \mathbf{E}[Z_n \mid Z_n > 0]$.

Evaluating these theorems and recollecting the terminology for Galton-Watson processes our results suggest the following manner of speaking. According to Theorem 1, (viii) we may in case of $q < 1$ distinguish the alternatives that $\lim_{n \rightarrow \infty} \mu_n$ is finite or infinite. The first one covers sticky processes and the second one the truly supercritical processes. In the case $q = 1$ we call the processes critical under the assumptions of Theorem 4 (then we necessarily have $\sum_{k=1}^{\infty} \frac{\nu_k}{\mu_{k-1}} = \infty$) and subcritical under the conditions of Theorem 3 (then necessarily $\lim_{n \rightarrow \infty} \mu_n = 0$). This results in the following classification of a branching process in environment $v = (f_1, f_2, \dots)$ under

assumption (A). Term it

$$\begin{aligned}
\text{supercritical, if} \quad & \lim_{n \rightarrow \infty} \mu_n = \infty \text{ and } \sum_{k=1}^{\infty} \frac{\nu_k}{\mu_{k-1}} < \infty , \\
\text{sticky, if} \quad & 0 < \lim_{n \rightarrow \infty} \mu_n < \infty \text{ and } \sum_{k=1}^{\infty} \frac{\nu_k}{\mu_{k-1}} < \infty , \\
\text{critical, if} \quad & \sum_{k=1}^{\infty} \frac{\nu_k}{\mu_{k-1}} = \infty \text{ and } \frac{1}{\mu_n} = o\left(\sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}}\right) , \\
\text{subcritical, if} \quad & \lim_{n \rightarrow \infty} \mu_n = 0 \text{ and } \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}} = O\left(\frac{1}{\mu_n}\right) .
\end{aligned}$$

There are also mixed cases which oszillate between the last two ‘pure’ regimes. Convergence of the means μ_n is not enforced in the critical case. They may diverge, converge to zero or even oszillate.

Examples.

(i) In the case $0 < \inf_n \nu_n \leq \sup_n \nu_n < \infty$ (as e.g. for Poisson variables) the classification simplifies. Here we are in the supercritical regime, iff $\sum_{k \geq 0} 1/\mu_k < \infty$ (which enforces $\mu_n \rightarrow \infty$). Roughly speaking this means that μ_n has to grow faster than linearly. On the other hand we are in the subcritical regime, iff $1/\mu_n \geq c \sum_{k=0}^{n-1} 1/\mu_k$ for some constant $c > 0$ (which enforces $\mu_n \rightarrow 0$). This implies $\mu_n \leq c^{-1}(1+c)^{1-n}$ for $n \geq 1$ (proof by induction), that is μ_n decreases at least at a geometric rate. Sticky behaviour is excluded, and there remains plenty of room for critical processes, that is for the processes which conform to the requirements $\sum_{k \geq 0} 1/\mu_k = \infty$ and $1/\mu_n = o(\sum_{k=0}^{n-1} 1/\mu_k)$.

(ii) In the *binary case* $\mathbf{P}(Y_n = 2) = p_n$, $\mathbf{P}(Y_n = 0) = 1 - p_n$ we get $f'_n(1) = \nu_n = 2p_n$ and $\mu_n = 2^n p_1 \cdots p_n$. This boils down to the same classification as in the previous example.

(iii) In the *symmetric case* $\mathbf{P}(Y_n = 0) = \mathbf{P}(Y_n = 2) = p_n/2$ and $\mathbf{P}(Y_n = 1) = 1 - p_n$ we have $\mu_n = 1$ and $\nu_n = p_n$. Here we find critical or sticky behaviour, according to whether $\sum_{k=1}^{\infty} p_n$ is divergent or convergent. \square

Our proofs use mainly tools from analysis. We are faced with the task to treat the probability measures $f_1 \circ \cdots \circ f_n$, which, as is well-known, are the distributions of the random variables Z_n . In order to handle such iterated compositions of generating functions we resort to a device which has been applied from the beginning in the theory of branching processes. For a probability distribution f on \mathbb{N}_0 with positive, finite mean m we define a function $\varphi : [0, 1) \rightarrow \mathbb{R}$ through the equation

$$\frac{1}{1-f(s)} = \frac{1}{m(1-s)} + \varphi(s) , \quad 0 \leq s < 1 .$$

To a certain extent the mean and the ‘shape’ of f are decoupled in this way. Indeed, Lemma 3 below shows that φ takes values which are of the magnitude of the standardized second factorial moment ν . Therefore we briefly call φ the *shape function* of f . As we shall see these functions are useful to dissolve the generating function $f_1 \circ \cdots \circ f_n$ into a sum. Here our contribution consists in obtaining sharp upper and lower bounds for the function φ and its derivative φ' , which then serve to precisely estimate the survival probabilities $\mathbf{P}(Z_n > 0)$.

Concluding this introduction let us comment on the literature. Agresti [1] already derived the sharp upper bound for φ which we give below. We shall see that this bound can be considered as a special case of the well-known Paley-Zygmund inequality. Agresti also obtained a lower bound for the survival probabilities, which, however, in general is away from our sharp bound. The contribution of Lyons [13] to Theorem 1 has been already mentioned, his methods are completely different from ours. He also proved Theorem 2, again under the assumption that the offspring numbers are a.s. uniformly bounded by a constant. D’Souza and Biggins [5] obtained Theorem 2 under a different set of assumptions. They require that there are numbers $a > 0, b > 1$ such that $\mu_{m+n}/\mu_m \geq ab^n$ for all $m, n \geq 1$ (called the uniform supercritical case). They do not need finite second moments but assume instead that the random variables Y_n are uniformly dominated by a random variable Y with $\mathbf{E}[Y \log^+ Y] < \infty$. Goettge [9] obtains $\mathbf{E}[W] = 1$ under the alleviated condition $\mu_n \geq an^b$ with $a > 0, b > 1$ (together with a uniform domination assumption), but doesn’t consider the validity of the equation $\mathbf{P}(W = 0) = q$. In order to prove the conditional limit law in Theorem 4 Jagers [10] draws attention to uniform estimates due to Sevast’yanov [15] (see also Lemma 3 in [6]). However, this approach demands amongst others the strong assumption that the sequence $\mathbf{E}[Z_n]$, $n \geq 0$, is bounded from above and away from zero. Independently and in parallel to our work N. Bhattacharya and M. Perlman [2] have presented a considerable generalization of Jager’s result, on a different route and under assumptions which are somewhat stronger than ours.

The paper is organized as follows. In Section 2 we discuss the assumptions and several examples. In Section 3 we analyze the shape function φ . Then Section 4 contains the proofs of our theorems. In the Appendix we return to Lindvall’s theorem.

2 Examples

In this section we determine constants $c < \infty$ such that for a \mathbb{N}_0 -valued random variable Y with a given distribution we have

$$\mathbf{E}[Y(Y-1)] \leq c\mathbf{E}[Y] \cdot \mathbf{E}[Y-1 \mid Y \geq 1] .$$

It is convenient to rewrite this inequality as

$$\mathbf{E}[Y(Y-1)] \leq c\mathbf{E}[Y \mid Y \geq 1] \cdot \mathbf{E}[(Y-1)^+] . \quad (1)$$

Here are two simple instances.

Examples.

- (i) Suppose that $Y \leq c$ a.s. for some constant $c < \infty$. Then because of $\mathbf{E}[Y \mid Y \geq 1] \geq 1$

$$\mathbf{E}[Y(Y-1)] = \mathbf{E}[Y(Y-1)^+] \leq c\mathbf{E}[(Y-1)^+] \leq c\mathbf{E}[Y \mid Y \geq 1]\mathbf{E}[(Y-1)^+] .$$

Thus Assumption (A) is satisfied, if the sequence (Y_n) is a.s. uniformly bounded.

(ii) Let Y have a *linear fractional distribution* that is

$$\mathbf{P}(Y = y \mid Y \geq 1) = (1 - p)^{y-1} p, \quad y \geq 1$$

with some $0 < p < 1$. This example covers geometric distributions. Then

$$\begin{aligned} \mathbf{E}[Y(Y-1)] &= \frac{2(1-p)}{p^2} \mathbf{P}(Y \geq 1), \quad \mathbf{E}[Y \mid Y \geq 1] = \frac{1}{p}, \\ \mathbf{E}[(Y-1)^+] &= \frac{(1-p)}{p} \mathbf{P}(Y \geq 1) \end{aligned}$$

implying that (1) holds with $c = 2$. □

The following proposition provides a useful tool for obtaining (1).

Proposition. *Let Y be a random variable with values in \mathbb{N}_0 and let $c < \infty$ be such that*

$$\mathbf{E}[Y(Y-1)(Y-2)] \leq c \mathbf{E}[Y(Y-1)](1 + \mathbf{E}[Y]) < \infty.$$

Then it follows

$$\mathbf{E}[Y(Y-1)] \leq (3c+2) \mathbf{E}[Y] \mathbf{E}[Y-1 \mid Y \geq 1].$$

In particular the Proposition shows that (A*) implies (A). Before proving it we discuss further examples.

Examples.

(iii) Let Y be *Poisson* with parameter $\lambda > 0$. Then

$$\mathbf{E}[Y(Y-1)(Y-2)] = \lambda^3 \leq \lambda^2(\lambda+1) = \mathbf{E}[Y(Y-1)](1 + \mathbf{E}[Y]).$$

For this type of distribution (A) is fulfilled with $c_1 = 5$.

(iv) For *binomial* Y with parameters $n \geq 1$ and $0 < p < 1$ the situation is the same, here

$$\mathbf{E}[Y(Y-1)(Y-2)] = n(n-1)(n-2)p^3 \leq n(n-1)p^2 np \leq \mathbf{E}[Y(Y-1)](1 + \mathbf{E}[Y]).$$

(v) For a *hypergeometric distribution* with parameter (N, K, n) we have for $N \geq 3$

$$\begin{aligned} \mathbf{E}[Y(Y-1)(Y-2)] &= \frac{n(n-1)(n-2)K(K-1)(K-2)}{N(N-1)(N-2)} \\ &\leq 3 \frac{n(n-1)K(K-1)}{N(N-1)} \frac{nK}{N} \leq 3 \mathbf{E}[Y(Y-1)](1 + \mathbf{E}[Y]), \end{aligned}$$

and (1) is satisfied with $c_1 = 11$. The case $N \leq 2$ can immediately be included.

(vi) For *negative binomial distributions* the generating function is given by

$$f(s) = \left(\frac{p}{1 - s(1 - p)} \right)^\alpha$$

with $0 < p < 1$ and a positive integer α . Now

$$\begin{aligned} \mathbf{E}[Y] &= \alpha \frac{1-p}{p}, \quad \mathbf{E}[Y(Y-1)] = \alpha(\alpha+1) \frac{(1-p)^2}{p^2}, \\ \mathbf{E}[Y(Y-1)(Y-2)] &= \alpha(\alpha+1)(\alpha+2) \frac{(1-p)^3}{p^3}. \end{aligned}$$

Thus

$$\mathbf{E}[Y(Y-1)(Y-2)] \leq 3\mathbf{E}[Y(Y-1)](1 + \mathbf{E}[Y]).$$

Again (A) is fulfilled with $c_1 = 11$. □

Now we come to the proof of the Proposition. It builds on the following Lemmas which will be useful later, too.

Lemma 1. *Let $g_1, g_2 \in \mathcal{P}$ have the same support and satisfy the following property: For any $y \in \mathbb{N}_0$ with $g_1[y] > 0$ it follows*

$$\frac{g_1[z]}{g_1[y]} \leq \frac{g_2[z]}{g_2[y]} \text{ for all } z > y.$$

Also let $\alpha : \mathbb{N}_0 \rightarrow \mathbb{R}$ be a non-decreasing function. Then

$$\sum_{y=0}^{\infty} \alpha(y) g_1[y] \leq \sum_{y=0}^{\infty} \alpha(y) g_2[y].$$

Proof. Let $y \geq 0$ such that $g_1[y] > 0$. Then $g_2[y] > 0$ and $\sum_{z \geq y} g_1[z]/g_1[y] \leq \sum_{z \geq y} g_2[z]/g_2[y]$. Also, if $g_1[z] > 0$ for some $z < y$, then by assumption we have $g_1[y]/g_1[z] \leq g_2[y]/g_2[z]$ and consequently $\sum_{z < y} g_1[z]/g_1[y] \geq \sum_{z < y} g_2[z]/g_2[y]$. Hence

$$\frac{\sum_{z \geq y} g_1[z]}{1 - \sum_{z \geq y} g_1[z]} = \frac{\sum_{z \geq y} g_1[z]/g_1[y]}{\sum_{z < y} g_1[z]/g_1[y]} \leq \frac{\sum_{z \geq y} g_2[z]/g_2[y]}{\sum_{z < y} g_2[z]/g_2[y]} = \frac{\sum_{z \geq y} g_2[z]}{1 - \sum_{z \geq y} g_2[z]}$$

and consequently

$$\sum_{z \geq y} g_1[z] \leq \sum_{z \geq y} g_2[z].$$

It follows that this inequality holds for all $y \geq 0$, since vanishing summands on the left-hand side may be removed.

Now define the function $\beta : \mathbb{N}_0 \rightarrow \mathbb{R}$ by $\beta(0) + \dots + \beta(y) = \alpha(y)$, $y \geq 0$. By assumption $\beta \geq 0$. It follows

$$\sum_{z=0}^{\infty} \alpha(z) g_1[z] = \sum_{y=0}^{\infty} \beta(y) \sum_{z=y}^{\infty} g_1[z] \leq \sum_{y=0}^{\infty} \beta(y) \sum_{z=y}^{\infty} g_2[z] = \sum_{z=0}^{\infty} \alpha(z) g_2[z].$$

This is our claim. □

Lemma 2. *If the \mathbb{N}_0 -valued random variable Y satisfies $0 < \mathbf{E}[Y(Y-1)] < \infty$, then*

$$\mathbf{E}[(Y-1)^+] \geq \frac{\mathbf{E}[Y(Y-1)]^2}{2\mathbf{E}[Y(Y-1)] + \mathbf{E}[Y(Y-1)(Y-2)]} \quad (2)$$

and

$$\frac{\mathbf{E}[Y(Y-1)]}{\mathbf{E}[Y]} \leq \frac{\mathbf{E}[Y(Y-1)(Y-2)]}{\mathbf{E}[Y(Y-1)]} + 1. \quad (3)$$

Proof. Let f be the distribution of Y and consider the probability measures g_1 and g_2 with weights

$$g_1[y] := \frac{(y-1)f[y]}{\sum_{z \geq 2} (z-1)f[z]}, \quad g_2[y] := \frac{y(y-1)f[y]}{\sum_{z \geq 2} z(z-1)f[z]}, \quad y \geq 2.$$

Because of $0 < \mathbf{E}[Y(Y-1)] < \infty$ they are well-defined, and they fulfil the assumption of Lemma 1, therefore it follows with $\alpha(y) := y$

$$\frac{\sum_{y \geq 2} y(y-1)f[y]}{\sum_{z \geq 2} (z-1)f[z]} \leq \frac{\sum_{y \geq 2} y^2(y-1)f[y]}{\sum_{z \geq 2} z(z-1)f[z]}.$$

Equivalently

$$\frac{\mathbf{E}[Y(Y-1)]}{\mathbf{E}[(Y-1)^+]} \leq \frac{\mathbf{E}[Y^2(Y-1)]}{\mathbf{E}[Y(Y-1)]},$$

which translates to our first claim.

If we choose instead

$$g_1[y] := \frac{yf[y]}{\sum_{z \geq 2} yf[z]}, \quad y \geq 2,$$

g_2 as above and $\alpha(y) = y-1$, then our argument yields the estimate

$$\frac{\mathbf{E}[Y(Y-1)]}{\mathbf{E}[Y; Y \geq 2]} \leq \frac{\mathbf{E}[Y(Y-1)^2]}{\mathbf{E}[Y(Y-1)]}.$$

This implies the second claim. □

Proof of the Proposition. In the case $\mathbf{E}[Y(Y-1)] = 0$ the assertion is obvious, thus we assume $0 < \mathbf{E}[Y(Y-1)] < \infty$. We distinguish two cases.

First let $\mathbf{E}[Y] \leq 2$. Then by assumption

$$\mathbf{E}[Y(Y-1)(Y-2)] \leq 3c\mathbf{E}[Y(Y-1)]$$

and consequently from (2)

$$\mathbf{E}[(Y-1)^+] \geq \frac{\mathbf{E}[Y(Y-1)]}{2+3c}.$$

Since $\mathbf{E}[Y \mid Y \geq 1] \geq 1$ we end up with

$$\mathbf{E}[Y(Y-1)] \leq (3c+2)\mathbf{E}[(Y-1)^+]\mathbf{E}[Y \mid Y \geq 1].$$

In view of (1) this is our claim.

Next let $\mathbf{E}[Y] \geq 2$. Then from (3) and our assumption

$$\frac{\mathbf{E}[Y(Y-1)]}{\mathbf{E}[Y]} \leq \frac{\mathbf{E}[Y(Y-1)(Y-2)]}{\mathbf{E}[Y(Y-1)]} + 1 \leq c([\mathbf{E}[Y] + 1) + 1 \leq \frac{3c+1}{2} \mathbf{E}[Y] .$$

Also

$$\mathbf{E}[(Y-1)^+] \geq \mathbf{E}[Y] - 1 \geq \frac{1}{2} \mathbf{E}[Y] , \quad \mathbf{E}[Y \mid Y \geq 1] \geq \mathbf{E}[Y] .$$

Hence

$$\mathbf{E}[Y(Y-1)] \leq \frac{3c+1}{2} \mathbf{E}[Y]^2 \leq (3c+1) \mathbf{E}[(Y-1)^+] \mathbf{E}[Y \mid Y \geq 1] .$$

This implies the claim. \square

3 Bounds for the shape function

For $f \in \mathcal{P}$ with mean $0 < m = f'(1) < \infty$ define the *shape function* $\varphi = \varphi_f : [0, 1] \rightarrow \mathbb{R}$ through the equation

$$\frac{1}{1-f(s)} = \frac{1}{m(1-s)} + \varphi(s) , \quad 0 \leq s < 1 .$$

Due to convexity of $f(s)$ we have $\varphi(s) \geq 0$ for all $0 \leq s < 1$. By means of a Taylor expansion of f around 1 one obtains $\lim_{s \uparrow 1} \varphi(s) = f''(1)/(2f'(1)^2)$, thus we extend φ by setting

$$\varphi(1) := \frac{1}{2} \frac{f''(1)}{f'(1)^2} = \frac{\nu}{2} .$$

In this section we prove the following sharp bounds.

Lemma 3. *Assume $f''(1) < \infty$. Then for $0 \leq s \leq 1$*

$$\frac{1}{2} \varphi(0) \leq \varphi(s) \leq 2\varphi(1) . \tag{4}$$

Note that φ is identical zero if $f[z] = 0$ for all $z \geq 2$. Else $\varphi(0) > 0$, and the lower bound of φ becomes strictly positive. Choosing $s = 1$ and $s = 0$ in (4) we obtain $\varphi(0)/2 \leq \varphi(1)$ and $\varphi(0) \leq 2\varphi(1)$. Note that for $f = \delta_k$ (Dirac-measure at point k) and $k \geq 2$ we have $\varphi(1) = \varphi(0)/2$ implying that the constants $1/2$ and 2 in (4) cannot be improved. The upper bound was derived in [8] using a different method of proof.

Lemma 4. *Assume $f'''(1) < \infty$. Then for $0 \leq s \leq 1$*

$$-\frac{f''(1)^2}{f'(1)^3} \leq \varphi'(s) \leq \frac{2}{3} \frac{f'''(1)}{f'(1)^2} + 2 \frac{f''(1)^2}{f'(1)^3}$$

These bounds are put together from right building blocks as can be seen from the formula

$$\varphi'(1) = \frac{1}{6} \frac{f'''(1)}{f'(1)^2} - \frac{1}{4} \frac{f''(1)^2}{f'(1)^3} ,$$

which follows by means of Taylor expansions of f and f' around 1. – Uniform estimates of $\varphi(1) - \varphi(s)$ have already been obtained by Sevast'yanov [15] and others (see Lemma 3 in [6]). Our lemma implies and generalizes these estimates.

Proof of Lemma 3. (i) First we examine a special case of Lemma 1. Consider for $0 < s \leq 1$ and $r \in \mathbb{N}_0$ the probability measures

$$g_s[y] = \frac{s^{r-y}}{1 + s + \dots + s^r}, \quad 0 \leq y \leq r.$$

Then for $0 < s \leq t \leq 1$, $0 \leq y < z \leq r$ we have $g_s[z]/g_s[y] = s^{y-z} \geq t^{y-z} = g_t[z]/g_t[y]$. We therefore obtain that

$$\sum_{y=0}^r y g_s[y] = \frac{s^{r-1} + 2s^{r-2} + \dots + r}{1 + s + \dots + s^r}$$

is a decreasing function in s . Also $\sum_{y=0}^r y g_0[y] = r$ and $\sum_{y=0}^r y g_1[y] = r/2$, and it follows for $0 \leq s \leq 1$

$$\frac{r}{2} \leq \frac{r + (r-1)s + \dots + s^{r-1}}{1 + s + \dots + s^r} \leq r. \quad (5)$$

(ii) Next we derive a second representation for φ . We have

$$1 - f(s) = \sum_{z=1}^{\infty} f[z](1 - s^z) = (1 - s) \sum_{z=1}^{\infty} f[z] \sum_{k=0}^{z-1} s^k,$$

and

$$\begin{aligned} f'(1)(1 - s) - (1 - f(s)) &= (1 - s) \sum_{z=1}^{\infty} f[z] \sum_{k=0}^{z-1} (1 - s^k) \\ &= (1 - s)^2 \sum_{z=1}^{\infty} f[z] \sum_{k=1}^{z-1} \sum_{j=0}^{k-1} s^j \\ &= (1 - s)^2 \sum_{z=1}^{\infty} f[z] ((z-1) + (z-2)s + \dots + s^{z-2}). \end{aligned}$$

Therefore

$$\begin{aligned} \varphi(s) &= \frac{f'(1)(1 - s) - (1 - f(s))}{f'(1)(1 - s)(1 - f(s))} \\ &= \frac{\sum_{y=1}^{\infty} f[y]((y-1) + (y-2)s + \dots + s^{y-2})}{f'(1) \cdot \sum_{z=1}^{\infty} f[z](1 + s + \dots + s^{z-1})}. \end{aligned}$$

From (5) it follows

$$\varphi(s) \leq \frac{\psi(s)}{f'(1)} \leq 2\varphi(s) \quad (6)$$

with

$$\psi(s) := \frac{\sum_{y=1}^{\infty} f[y](y-1)(1 + s + \dots + s^{y-1})}{\sum_{z=1}^{\infty} f[z](1 + s + \dots + s^{z-1})}.$$

Now consider the probability measures $g_s \in \mathcal{P}$, $0 \leq s \leq 1$, given by

$$g_s[y] := \frac{f[y](1 + s + \dots + s^{y-1})}{\sum_{z=1}^{\infty} f[z](1 + s + \dots + s^{z-1})}, \quad y \geq 1. \quad (7)$$

Then for $f[y] > 0$ and $z > y$, after some algebra,

$$\frac{g_s[z]}{g_s[y]} = \frac{f[z]}{f[y]} \prod_{v=1}^{z-y} \left(1 + \frac{1}{s^{-1} + \dots + s^{-y-v+1}} \right),$$

which is an increasing function in s . Therefore by Lemma 1 the function $\psi(s)$ is increasing in s . In combination with (6) we get

$$\varphi(s) \leq \frac{\psi(s)}{f'(1)} \leq \frac{\psi(1)}{f'(1)} \leq 2\varphi(1), \quad 2\varphi(s) \geq \frac{\psi(s)}{f'(1)} \geq \frac{\psi(0)}{f'(1)} \geq \varphi(0).$$

This gives the claim of the proposition. \square

Proof of Lemma 4. We have

$$m\varphi'(s) = \frac{mf'(s)}{(1-f(s))^2} - \frac{1}{(1-s)^2}.$$

In this formula we replace the geometric mean $\sqrt{mf'(s)}$ by the arithmetic means $(m + f'(s))/2$ leading to the formula

$$m\varphi'(s) = \psi_1(s) - \psi_2(s) \quad (8)$$

with

$$\psi_1(s) = \frac{1}{4} \frac{(m + f'(s))^2}{(1-f(s))^2} - \frac{1}{(1-s)^2}, \quad \psi_2(s) = \frac{1}{4} \frac{(m + f'(s))^2}{(1-f(s))^2} - \frac{mf'(s)}{(1-f(s))^2}.$$

We show that ψ_1 and ψ_2 both are non-negative functions and estimate them from above.

To accomplish this for ψ_1 we write

$$\begin{aligned} & \frac{1}{2}(m + f'(s))(1-s) - (1-f(s)) \\ &= \frac{1-s}{2} \sum_{y=1}^{\infty} y(1+s^{y-1})f[y] - \sum_{y=1}^{\infty} (1-s^y)f[y] \\ &= \frac{1-s}{2} \sum_{y=3}^{\infty} \left(y(1+s^{y-1}) - 2(1+s+\dots+s^{y-1}) \right) f[y] \\ &= \frac{1-s}{2} \zeta(s) \quad (\text{say}). \end{aligned}$$

Since

$$\begin{aligned} & \frac{d}{ds} (y(1+s^{y-1}) - 2(1+s+\dots+s^{y-1})) \\ &= y(y-1)s^{y-2} - 2(1+2s+\dots+(y-1)s^{y-2}) \\ &\leq y(y-1)s^{y-2} - 2s^{y-2}(1+2+\dots+(y-1)) = 0 \end{aligned}$$

for all $0 \leq s \leq 1$, and since $\zeta(1) = 0$ we see that ζ is a non-negative, decreasing function. Thus ψ_1 is a non-negative function, too. Moreover we have for $y \geq 3$ the polynomial identity (check it!)

$$y(1 + s^{y-1}) - 2(1 + s + \cdots + s^{y-1}) = (1 - s)^2 \sum_{z=1}^{y-2} z(y - z - 1)s^{z-1} ,$$

and consequently

$$\zeta(s) = (1 - s)^2 \sum_{y=3}^{\infty} \sum_{z=1}^{y-2} z(y - z - 1)s^{z-1} f[y] = (1 - s)^2 \xi(s) \quad (\text{say}) ,$$

where ξ is a non-negative, increasing function.

Coming back to

$$\psi_1(s) = \frac{\frac{1}{2}(m + f'(s))(1 - s) - (1 - f(s))}{(1 - f(s))(1 - s)} \cdot \frac{\frac{1}{2}(m + f'(s))(1 - s) + (1 - f(s))}{(1 - f(s))(1 - s)}$$

it follows

$$\begin{aligned} \psi_1(s) &\leq \frac{\zeta(s)}{2(1 - f(s))} \left(\frac{m}{1 - f(s)} + \frac{1}{1 - s} \right) \\ &= \frac{\zeta(s)}{2} \left(\frac{1}{m(1 - s)} + \varphi(s) \right) \left(\frac{2}{1 - s} + m\varphi(s) \right) \\ &\leq \frac{2\zeta(s)}{m} \left(\frac{1}{(1 - s)^2} + m^2\varphi(s)^2 \right) \\ &= \frac{2\xi(s)}{m} + 2m\zeta(s)\varphi(s)^2 . \end{aligned}$$

Using Lemma 3 and the monotonicity properties of ξ and ζ we obtain

$$\psi_1(s) \leq \frac{2\xi(1)}{m} + 8m\zeta(0)\varphi(1)^2 .$$

Also $\zeta(0) \leq m$ and

$$\xi(1) = \sum_{y=3}^{\infty} \sum_{z=1}^{y-2} z(y - z - 1)f[z] = \frac{1}{3} \sum_{y=3}^{\infty} z(z - 1)(z - 2)f[z] = \frac{f'''(1)}{3} .$$

Altogether this amount to

$$0 \leq \psi_1(s) \leq \frac{2f'''(1)}{3f'(1)} + 2\frac{f''(1)^2}{f'(1)^2} . \quad (9)$$

Now we investigate the function ψ_2 , which we rewrite as

$$\psi_2(s) = \frac{1}{4} \left(\frac{m - f'(s)}{1 - f(s)} \right)^2 .$$

We have

$$1 - f(s) = \sum_{z=1}^{\infty} (1 - s^z) f[z] = (1 - s) \sum_{z=1}^{\infty} (1 + s + \cdots + s^{z-1}) f[z]$$

and

$$m - f'(s) = \sum_{y=1}^{\infty} (1 - s^{y-1}) y f[y] = (1 - s) \sum_{y=2}^{\infty} y (1 + \cdots + s^{y-2}) f[y] .$$

Using the notation (7) it follows

$$\frac{m - f'(s)}{1 - f(s)} = \sum_{y=2}^{\infty} \frac{1 + \cdots + s^{y-2}}{1 + \cdots + s^{y-1}} y g_s[y] \leq \sum_{y=2}^{\infty} y g_s[y] .$$

As above we may apply Lemma 1 to the probability measures g_s and conclude that the right-hand term is increasing with s . Therefore

$$0 \leq \frac{m - f'(s)}{1 - f(s)} \leq \sum_{y=2}^{\infty} y g_1[y] = \frac{\sum_{y=2}^{\infty} y^2 f[y]}{\sum_{z=1}^{\infty} z f[z]} \leq \frac{2 \sum_{y=1}^{\infty} y(y-1) f[y]}{\sum_{z=1}^{\infty} z f[z]} = \frac{2 f''(1)}{f'(1)}$$

and hence

$$0 \leq \psi_2(s) \leq \frac{f''(1)^2}{f'(1)^2} .$$

Combining this estimate with (8) and (9) our claim follows. \square

4 Proof of the theorems

Let $v = (f_1, f_2, \dots)$ denote a varying environment. Let us define for non-negative integers $k \leq n$ the probability measures

$$f_{k,n} := f_{k+1} \circ \cdots \circ f_n$$

with the convention $f_{n,n} = \delta_1$ (the dirac measure at point 1). As is well-known, the distribution of Z_n is given by $f_{0,n}$. Thus for a BPVE one is faced with the task to analyze such probability measures.

First let us review some formulas for moments. There exists a clear-cut expression for the variance of Z_n due to Fearn [7]. It seems to be less noticed that there is a similar appealing formula for the second factorial moment of Z_n , which turns out to be more useful for our purpose.

Lemma 5. *For a BPVE $(Z_n)_{n \geq 0}$ we have*

$$\mathbf{E}[Z_n] = \mu_n , \quad \frac{\mathbf{E}[Z_n(Z_n - 1)]}{\mathbf{E}[Z_n]^2} = \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}} .$$

The proof is standard. We have

$$f'_{k,n}(s) = \prod_{l=k+1}^n f'_l(f_{l,n}(s)) ,$$

in particular $f'_{n,n}(s) = 1$, and after some rearrangements

$$f''_{k,n}(s) = f'_{k,n}(s)^2 \sum_{l=k+1}^n \frac{f''_l(f_{l,n}(s))}{f'_l(f_{l,n}(s))^2 \prod_{j=k+1}^{l-1} f'_j(f_{j,n}(s))} ,$$

in particular $f''_{n,n}(s) = 0$. Choosing $k = 0$ and $s = 1$ the Lemma 5 is proved.

Next we recall an expansion of the generating function of Z_n taken from [11] and [8]. It is a kind of formula which has been used in many studies of branching processes. Let φ_n , $n \geq 1$, be the shape functions of f_n , $n \geq 1$. Then, since $f_{k,n} = f_{k+1} \circ f_{k+1,n}$ for $k < n$,

$$\frac{1}{1 - f_{k,n}(s)} = \frac{1}{f'_{k+1}(1)(1 - f_{k+1,n}(s))} + \varphi_1(f_{k+1,n}(s)) .$$

Iterating the formula we end up with the following identity.

Lemma 6. For $0 \leq s < 1$, $0 \leq k < n$

$$\frac{1}{1 - f_{k,n}(s)} = \frac{\mu_k}{\mu_n(1 - s)} + \mu_k \sum_{l=k+1}^n \frac{\varphi_l(f_{l,n}(s))}{\mu_{l-1}} .$$

The next lemma clarifies the role of Assumption (A).

Lemma 7. Under Assumption (A) there is a $c < \infty$ such that for all $n \geq 0$

$$\frac{\mathbf{E}[Z_n]^2}{\mathbf{E}[Z_n^2]} \leq \mathbf{P}(Z_n > 0) \leq c \frac{\mathbf{E}[Z_n]^2}{\mathbf{E}[Z_n^2]} .$$

Proof. The left-hand estimate is just the Paley-Zygmund inequality. For the right-hand estimate observe that $\mathbf{P}(Z_n > 0) = 1 - f_{0,n}[0] = 1 - f_{0,n}(0)$. Using Lemma 6 with $s = 0$ we get the representation

$$\frac{1}{\mathbf{P}(Z_n > 0)} = \frac{1}{\mu_n} + \sum_{k=1}^n \frac{\varphi_k(f_{k,n}(0))}{\mu_{k-1}} , \quad (10)$$

hence by means of Lemma 3

$$\frac{1}{\mathbf{P}(Z_n > 0)} \geq \frac{1}{\mu_n} + \frac{1}{2} \sum_{k=1}^n \frac{\varphi_k(0)}{\mu_{k-1}} . \quad (11)$$

Now

$$\varphi_k(0) = \frac{1}{1 - f_k[0]} - \frac{1}{f'_k(1)} = \frac{\mathbf{E}[(Y_k - 1)^+]}{\mathbf{P}(Y_k > 0)\mathbf{E}[Y_k]} ,$$

such that (A) is in view of (1) equivalent to the statement

$$\varphi_k(0) \geq \frac{1}{c_1} \frac{\mathbf{E}[Y_k(Y_k - 1)]}{\mathbf{E}[Y_k]^2} = \frac{\nu_k}{c_1} . \quad (12)$$

It follows with $c = \max(1, 2c_1)$

$$\frac{1}{\mathbf{P}(Z_n > 0)} \geq \frac{1}{\mu_n} + \frac{1}{2c_1} \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}} \geq \frac{1}{c} \left(\frac{1}{\mu_n} + \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}} \right).$$

On the other hand Lemma 5 implies

$$\frac{\mathbf{E}[Z_n^2]}{\mathbf{E}[Z_n]^2} = \frac{\mathbf{E}[Z_n(Z_n - 1)]}{\mathbf{E}[Z_n]^2} + \frac{1}{\mathbf{E}[Z_n]} = \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}} + \frac{1}{\mu_n}. \quad (13)$$

Combining the last two formulas our claim follows. \square

Proof of Theorem 1. (i) \Leftrightarrow (ii): Since $\lim_{n \rightarrow \infty} \mathbf{P}(Z_n > 0) = 1 - q$ the equivalence follows from Lemma 7.

(ii) \Leftrightarrow (iii): We have

$$\begin{aligned} \sum_{k=1}^n \frac{\rho_k}{\mu_{k-1}} &= \sum_{k=1}^n \frac{\nu_k + f_k(1)^{-1} - 1}{\mu_{k-1}} \\ &= \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}} + \sum_{k=1}^n \left(\frac{1}{\mu_k} - \frac{1}{\mu_{k-1}} \right) = \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}} + \frac{1}{\mu_n} - 1, \end{aligned} \quad (14)$$

thus because of (13)

$$\frac{\mathbf{E}[Z_n^2]}{\mathbf{E}[Z_n]^2} = \sum_{k=1}^n \frac{\rho_k}{\mu_{k-1}} + 1. \quad (15)$$

This gives the claim.

(iii) \Leftrightarrow (iv): This equivalence is an immediate consequence of (14).

(v) \Leftrightarrow (vi): This implication follows again from Lemma 7.

(vi) \Leftrightarrow (vii): Again this is a consequence of equation (15).

(vii) \Leftrightarrow (viii): This claim follows from (14). \square

Remark. From (11) it follows that a sufficient condition for a.s. extinction is given by the single requirement $\sum_{k \geq 1} \varphi_k(0)/\mu_{k-1} = \infty$. This confirms a conjecture of Jirina [11]. \square

Proof of Theorem 2. Statement (i) is obvious. For the first part of statement (ii) note that from Theorem 1, (vi) it follows that $\sup_{n \geq 0} \mathbf{E}[W_n^2] < \infty$. Therefore the martingale $(W_n)_{n \geq 0}$ is square-integrable implying $\mathbf{E}[W] = \mathbf{E}[W_0] = 1$.

For the other part we distinguish two cases. Either $\mu_n \rightarrow r$ with $0 < r < \infty$. Then $W_n = Z_n/\mu_n \rightarrow Z_\infty/r$ a.s., consequently $W = Z_\infty/r$ a.s. and $\mathbf{P}(W = 0) = \mathbf{P}(Z_\infty = 0) = q$. Else we may assume $\mu_n \rightarrow \infty$ in view of Theorem 1, (viii). Also $\{Z_\infty = 0\} \subset \{W = 0\}$ a.s., thus it is sufficient to show that $\mathbf{P}(Z_\infty > 0, W = 0) = 0$. First we estimate $\mathbf{P}(Z_\infty = 0 \mid Z_k = 1)$ from below. From Lemma 6 and Lemma 3 for $k < n$

$$\frac{1}{1 - \mathbf{P}(Z_n = 0 \mid Z_k = 1)} = \frac{1}{1 - f_{k,n}(0)} \geq \frac{1}{2} \mu_k \sum_{l=k+1}^n \frac{\varphi_l(0)}{\mu_{l-1}}.$$

as well as

$$\begin{aligned} \frac{1}{1 - \mathbf{E}[e^{-\lambda W_n} \mid Z_k = 1]} &= \frac{1}{1 - f_{k,n}(e^{-\lambda/\mu_n})} \\ &\leq \frac{\mu_k}{\mu_n(1 - e^{-\lambda/\mu_n})} + 2\mu_k \sum_{l=k+1}^n \frac{\varphi_l(1)}{\mu_{l-1}} \end{aligned}$$

with $\lambda > 0$. By means of $\varphi_l(1) = \nu_l/2$ and (12) this entails

$$\frac{1}{1 - \mathbf{E}[e^{-\lambda W_n} \mid Z_k = 1]} \leq \frac{\mu_k}{\mu_n(1 - e^{-\lambda/\mu_n})} + \frac{2c_1}{1 - \mathbf{P}(Z_n = 0 \mid Z_k = 1)} .$$

Letting $n \rightarrow \infty$ we get

$$\frac{1}{1 - \mathbf{E}[e^{-\lambda W} \mid Z_k = 1]} \leq \frac{\mu_k}{\lambda} + \frac{2c_1}{1 - \mathbf{P}(Z_\infty = 0 \mid Z_k = 1)}$$

and with $\lambda \rightarrow \infty$

$$\frac{1}{\mathbf{P}(W > 0 \mid Z_k = 1)} \leq \frac{2c_1}{\mathbf{P}(Z_\infty > 0 \mid Z_k = 1)} .$$

Using $e^{-2x} \leq 1 - x$ for $0 \leq x \leq 1/2$ it follows for $\mathbf{P}(W > 0 \mid Z_k = 1) \leq (4c_1)^{-1}$

$$\begin{aligned} \mathbf{P}(Z_\infty = 0 \mid Z_k = 1) &= 1 - \mathbf{P}(Z_\infty > 0 \mid Z_k = 1) \geq 1 - 2c_1 \mathbf{P}(W > 0 \mid Z_k = 1) \\ &\geq e^{-4c_1 \mathbf{P}(W > 0 \mid Z_k = 1)} \geq (1 - \mathbf{P}(W > 0 \mid Z_k = 1))^{4c_1} \\ &= \mathbf{P}(W = 0 \mid Z_k = 1)^{4c_1} . \end{aligned} \tag{16}$$

Now we draw on a martingale, which already appears in the work of D'Souza and Biggins [5]. Let for $n \geq 0$

$$M_n := \mathbf{P}(W = 0 \mid Z_0, \dots, Z_n) = \mathbf{P}(W = 0 \mid Z_n = 1)^{Z_n} \text{ a.s. .}$$

From standard martingale theory $M_n \rightarrow I\{W = 0\}$ a.s. In particular we have

$$\mathbf{P}(W = 0 \mid Z_n = 1)^{Z_n} \rightarrow 1 \text{ a.s. on the event that } W = 0 , \tag{17}$$

a result which has already been noticed and exploited by D'Souza [4].

We distinguish two cases. Either there is an infinite sequence of natural numbers such that $\mathbf{P}(W > 0 \mid Z_n = 1) > (4c_1)^{-1}$ along this sequence. Then (17) implies that $Z_n \rightarrow 0$ a.s. on the event $W = 0$. Or else we may apply our estimate (16) to obtain from (17) that

$$\mathbf{P}(Z_\infty = 0 \mid Z_n = 1)^{Z_n} \rightarrow 1 \text{ a.s. on the event that } W = 0 .$$

Therefore, given $\varepsilon > 0$, we have for n sufficiently large

$$\begin{aligned} \mathbf{P}(Z_\infty > 0, W = 0) &\leq \varepsilon + \mathbf{P}(Z_n > 0, \mathbf{P}(Z_\infty = 0 \mid Z_n = 1)^{Z_n} \geq 1 - \varepsilon) \\ &\leq \varepsilon + \frac{1}{1 - \varepsilon} \mathbf{E}[\mathbf{P}(Z_\infty = 0 \mid Z_n); Z_n > 0] \\ &= \varepsilon + \frac{1}{1 - \varepsilon} \mathbf{P}(Z_\infty = 0, Z_n > 0) . \end{aligned}$$

Letting $n \rightarrow \infty$ we obtain $\mathbf{P}(Z_\infty > 0, W = 0) \leq \varepsilon$, and the claim follows with $\varepsilon \rightarrow 0$. \square

Proof of Theorem 3. (i) \Rightarrow (iii): From Lemma 6, Lemma 3 and (12) we have for $0 \leq s < 1$

$$\mathbf{E}[1 - s^{Z_n} \mid Z_n > 0] = \frac{1 - f_{0,n}(s)}{1 - f_{0,n}(0)} \geq \frac{\frac{1}{2c_1} \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}}}{\frac{1}{\mu_n(1-s)} + \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}}}.$$

By assumption we may choose $s < 1$ such that the left-hand side is smaller than $1/(4c_1)$ for all $n \geq 0$ which implies

$$\frac{1}{\mu_n(1-s)} \geq \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}}.$$

Thus the implication is verified.

(iii) \Rightarrow (ii): From the left-hand inequality in Lemma 7 and from (13) we obtain

$$1 \leq \mathbf{E}[Z_n \mid Z_n > 0] = \frac{\mu_n}{\mathbf{P}(Z_n > 0)} \leq 1 + \mu_n \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}}$$

from which the claim follows.

(ii) \Rightarrow (i): This implication is obvious. □

The next lemma prepares the proof of Theorem 4. It clarifies the role of (A*).

Lemma 8. *Under the assumptions of Theorem 4 we have*

$$\sup_{0 \leq s \leq 1} \left| \sum_{k=1}^n \frac{\varphi_k(f_{k,n}(s))}{\mu_{k-1}} - \frac{1}{2} \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}} \right| = o\left(\sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}}\right)$$

as $n \rightarrow \infty$.

Proof. Recall that (A*) entails (A). Observe that the other assumptions of the Theorem 4 together with Theorem 1, (iii) imply

$$\sum_{k=1}^{\infty} \frac{\nu_k}{\mu_{k-1}} = \infty.$$

Denote $\psi_k := \sup_{0 \leq s \leq 1} |\varphi'_k(s)|$ and let $1 \leq r \leq n$. Because of Lemma 3 we have

$$\left| \sum_{k=1}^n \frac{\varphi_k(1)}{\mu_{k-1}} - \sum_{k=1}^n \frac{\varphi_k(f_{k,n}(s))}{\mu_{k-1}} \right| \leq \sum_{k=1}^{r-1} \frac{\psi_k}{\mu_{k-1}} (1 - f_{k,n}(0)) + 3 \sum_{k=r}^n \frac{\varphi_k(1)}{\mu_{k-1}}.$$

From Lemma 4 and formulas (3) and (A*) we obtain

$$\frac{\psi_k}{\nu_k} \leq \frac{2}{3} \frac{f_k'''(1)}{f_k''(1)} + 2 \frac{f_k''(1)}{f_k'(1)} \leq 3 \frac{f_k'''(1)}{f_k''(1)} + 2 \leq c(f_k'(1) + 1)$$

with $c = 3c_2 + 2$. Further from Lemma 6, Lemma 3 and (A) respectively (12)

$$\frac{1}{1 - f_{k,n}(0)} \geq \mu_k \sum_{l=k+1}^n \frac{\varphi_l(f_{l,n}(0))}{\mu_{l-1}} \geq \frac{\mu_k}{2} \sum_{l=k+1}^n \frac{\varphi_l(0)}{\mu_{l-1}} \geq \frac{\mu_k}{2c_1} \sum_{l=k+1}^n \frac{\nu_l}{\mu_{l-1}}.$$

These estimates together with $\varphi_k(1) = \nu_k/2$ imply

$$\begin{aligned} \left| \sum_{k=1}^n \frac{\varphi_k(f_{k,n}(s))}{\mu_{k-1}} - \frac{1}{2} \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}} \right| \\ \leq \bar{c} \sum_{k=1}^{r-1} \frac{\nu_k(1 + f'_k(1))}{\mu_{k-1}} \cdot \frac{1}{\mu_k \sum_{l=k+1}^n \frac{\nu_l}{\mu_{l-1}}} + \frac{3}{2} \sum_{k=r}^n \frac{\nu_k}{\mu_{k-1}} \end{aligned}$$

with $\bar{c} = 2cc_1$. Now let $\varepsilon > 0$ and define

$$r = r_{\varepsilon,n} = \min \left\{ 1 \leq k < n : \mu_k \sum_{l=k+1}^n \frac{\nu_l}{\mu_{l-1}} \leq \frac{2\bar{c}}{\varepsilon} (1 + f'_k(1)) \right\},$$

where we put $r = n$, if no $k < n$ fulfils the right-hand inequality. Then

$$\begin{aligned} \left| \sum_{k=1}^n \frac{\varphi_k(f_{k,n}(s))}{\mu_{k-1}} - \frac{1}{2} \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}} \right| &\leq \frac{\varepsilon}{2} \sum_{k=1}^{r-1} \frac{\nu_k}{\mu_{k-1}} + \frac{3}{2} \sum_{k=r}^n \frac{\nu_k}{\mu_{k-1}} \\ &\leq \frac{\varepsilon}{2} \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}} + \frac{3\nu_r}{2\mu_{r-1}} + \frac{3\bar{c}}{\varepsilon} \frac{1 + f'_r(1)}{\mu_r}. \end{aligned}$$

Moreover, using (3) and (A*),

$$\nu_r = \frac{f''_r(1)}{f'_r(1)^2} \leq \frac{1}{f'_r(1)} \left(\frac{f'''_r(1)}{f''_r(1)} + 1 \right) \leq (c_2 + 1) \left(1 + \frac{1}{f'_r(1)} \right).$$

Hence, since $\mu_r = f'_r(1)\mu_{r-1}$, there is a constant $c_\varepsilon > 0$ depending on ε such that

$$\left| \sum_{k=1}^n \frac{\varphi_k(f_{k,n}(s))}{\mu_{k-1}} - \frac{1}{2} \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}} \right| \leq \frac{\varepsilon}{2} \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}} + \frac{c_\varepsilon}{\mu_{r-1}} + \frac{c_\varepsilon}{\mu_r}.$$

In view of the lemma's assumption there is a positive integer r_ε such that for all $n \geq r > r_\varepsilon$

$$\frac{c_\varepsilon}{\mu_{r-1}} + \frac{c_\varepsilon}{\mu_r} \leq \frac{\varepsilon}{4} \sum_{k=1}^{r-1} \frac{\nu_k}{\mu_{k-1}} + \frac{\varepsilon}{4} \sum_{k=1}^r \frac{\nu_k}{\mu_{k-1}} \leq \frac{\varepsilon}{2} \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}}.$$

Since the right-hand term diverges as $n \rightarrow \infty$ it follows

$$\frac{c_\varepsilon}{\mu_{r-1}} + \frac{c_\varepsilon}{\mu_r} \leq \frac{\varepsilon}{2} \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}}$$

for all $0 < r \leq n$, if only n is large enough, and we obtain

$$\left| \sum_{k=1}^n \frac{\varphi_k(f_{k,n}(s))}{\mu_{k-1}} - \frac{1}{2} \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}} \right| \leq \varepsilon \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}}.$$

This proves our claim. \square

Proof of Theorem 4. From (10), Lemma 8 and the theorem's assumption it follows

$$\frac{1}{\mathbf{P}(Z_n > 0)} = \frac{1}{\mu_n} + \sum_{k=1}^n \frac{\varphi_k(f_{k,n}(0))}{\mu_{k-1}} \sim \frac{1}{2} \sum_{k=1}^n \frac{\nu_k}{\mu_{k-1}}$$

implying the first claim. Also, using Lemma 6 we have

$$\begin{aligned} 1 - \mathbf{E}[e^{-\lambda Z_n/a_n} \mid Z_n > 0] &= \frac{1 - f_{0,n}(e^{-\lambda/a_n})}{1 - f_{0,n}(0)} \\ &= \frac{\frac{1}{\mu_n} + \sum_{k=1}^n \frac{\varphi_k(f_{k,n}(0))}{\mu_{k-1}}}{\frac{1}{\mu_n(1 - e^{-\lambda/a_n})} + \sum_{k=1}^n \frac{\varphi_k(f_{k,n}(e^{-\lambda/a_n}))}{\mu_{k-1}}} . \end{aligned}$$

Since $a_n \rightarrow \infty$, from Lemma 8 and the theorem's assumption

$$1 - \mathbf{E}[e^{-\lambda Z_n/a_n} \mid Z_n > 0] = \frac{(1 + o(1)) \sum_{k=1}^n \frac{\nu_k}{2\mu_{k-1}}}{(1 + o(1)) \frac{a_n}{\lambda \mu_n} + (1 + o(1)) \sum_{k=1}^n \frac{\nu_k}{2\mu_{k-1}}}$$

as $n \rightarrow \infty$. From the definition of a_n we get

$$1 - \mathbf{E}[e^{-\lambda Z_n/a_n} \mid Z_n > 0] = \frac{\lambda + o(1)}{1 + \lambda} .$$

This implies the claim. \square

5 Appendix

Here we consider Lindvall's theorem [12]. His approach rests on the extensive calculations of Church [3]. We give a self-contained proof streamlining their ideas.

Theorem. *For a BPVE $(Z_n)_{n \geq 0}$ in varying environment $v = (f_1, f_2, \dots)$ there exists a random variable Z_∞ with values in $\mathbb{N}_0 \cup \{\infty\}$ such that as $n \rightarrow \infty$*

$$Z_n \rightarrow Z_\infty \text{ a.s.}$$

Moreover,

$$\mathbf{P}(Z_\infty = 0 \text{ or } \infty) = 1 \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} (1 - f_n[1]) = \infty .$$

Proof. (i) We prepare the proof by showing that the sequence of probability measures $f_{0,n}$ is vaguely converging to a (possibly defective) measure g on \mathbb{N}_0 . Note that $f_{0,n}[0] \rightarrow q$. Thus either $f_{0,n} \rightarrow q\delta_0$ vaguely (with the Dirac measure δ_0 at point 0), or else (by the Helly-Bray Theorem) there exists a sequence of integers $0 = n_0 < n_1 < n_2 < \dots$ such that, as $k \rightarrow \infty$, we have $f_{0,n_k} \rightarrow g$ vaguely with $g \neq q\delta_0$.

In the latter case the limiting generating function $g(s)$ is strictly increasing in s , and $f_{0,n_k}(s) \rightarrow g(s)$ for all $0 \leq s < 1$. Then, for given $n \in \mathbb{N}_0$, we define $l_n := n_k, m_n := n_{k+1}$ with $n_k \leq n < n_{k+1}$, thus $l_n \leq n < m_n$. We like to show that $f_{l_n,n}$ converges vaguely to δ_1 . For this purpose we consider

a subsequence n' such that both $f_{l_{n'},n'}$ and $f_{n',m_{n'}}$ converge vaguely to measures h_1 and h_2 . Going in $f_{0,m_{n'}} = f_{0,l_{n'}} \circ f_{l_{n'},n'} \circ f_{n',m_{n'}}$ to the limit we obtain

$$g(s) = g(h_1(h_2(s))) , \quad 0 \leq s < 1 .$$

Since g is strictly increasing, $h_1(h_2(s)) = s$, which for generating functions implies $h_1(s) = h_2(s) = s$. Thus indeed, using the common sub-sub-sequence argument, $f_{l_n,n} \rightarrow \delta_1$ as $n \rightarrow \infty$. It follows that, as $n \rightarrow \infty$

$$f_{0,n}(s) = f_{0,l_n}(f_{l_n,n}(s)) \rightarrow g(s) , \quad 0 \leq s < 1 ,$$

which means $f_{0,n} \rightarrow g$ vaguely, as has been claimed.

(ii) We now turn to the proof of the first statement. The case $g(s) = 1$ for all $0 \leq s < 1$ is obvious, then $g = \delta_0$ and $q = 1$, and Z_n is a.s. convergent to 0. Thus we are left with the case $g(s) < 1$ for all $s < 1$. Then there is a decreasing sequence $(b_n)_{n \geq 0}$ of real numbers, such that $f_{0,n}(1/2) \leq b_n \leq 1$ and $b_n \downarrow g(1/2)$. Define the sequence $(a_n)_{n \geq 0}$ through the equation

$$f_{0,n}(a_n) = b_n .$$

Therefore $1/2 \leq a_n \leq 1$, also we have $f_{0,n+1}(a_{n+1}) \leq f_{0,n}(a_n)$ or equivalently $f_{n+1}(a_{n+1}) \leq a_n$. Then the process $U = (U_n)_{n \geq 0}$, given by

$$U_n := a_n^{Z_n} \cdot 1_{\{Z_n > 0\}}$$

is a non-negative supermartingale. Indeed, because of $f_{n+1}(0)^{Z_n} \geq 1_{\{Z_n=0\}}$ and $f_{n+1}(a_{n+1}) \leq a_n$ we have

$$\mathbf{E}[U_{n+1} \mid Z_0, \dots, Z_n] = f_{n+1}(a_{n+1})^{Z_n} - f_{n+1}(0)^{Z_n} \leq a_n^{Z_n} - 1_{\{Z_n=0\}} = U_n \text{ a.s.}$$

Thus U_n is a.s. convergent to a random variable $U \geq 0$.

Now we distinguish two cases. Either $g \neq q\delta_0$. Then $g(s)$ is strictly increasing, which implies $a_n \rightarrow 1/2$ as $n \rightarrow \infty$. Hence the a.s. convergence of U_n enforces the a.s. convergence of Z_n with possible limit ∞ .

Or else $g = q\delta_0$. Then $g(1/2) = q$, implying that for $n \rightarrow \infty$

$$\mathbf{E}[U_n] = f_{0,n}(a_n) - f_{0,n}(0) = b_n - \mathbb{P}(Z_n = 0) \rightarrow g(1/2) - q = 0$$

and consequently $U = 0$ a.s. implying $U_n \rightarrow 0$ a.s. Since $a_n \geq 1/2$ for all n , this enforces that Z_n converges a.s. to 0 or ∞ . In both cases $Z_n \rightarrow Z_\infty$ a.s. for some random variable Z_∞ .

(iii) For the second statement we use the representation $Z_n = \sum_{i=1}^{Z_n-1} Y_{i,n}$. Define the events $A_{z,n} := \{\sum_{i=1}^z Y_{i,n} \neq z\}$. Then for $z \geq 1$

$$\mathbf{P}(A_{z,n}) \geq 3^{-z}(1 - f_n[1]) .$$

Indeed, if $f_n[1] \geq 1/3$, then

$$\begin{aligned} \mathbf{P}(A_{z,n}) &\geq \mathbf{P}(Y_{1,n} \neq 1, Y_{2,n} = \dots = Y_{z,n} = 1) \\ &\geq (1 - f_n[1])f_n[1]^z \geq 3^{-z}(1 - f_n[1]) , \end{aligned}$$

and if $f_n[1] \leq 1/3$, then either $\mathbf{P}(Y_{i,n} > 1) \geq 1/3$ or $\mathbf{P}(Y_{i,n} = 0) \geq 1/3$ implying

$$\begin{aligned}\mathbf{P}(A_{z,n}) &\geq \mathbf{P}(\min(Y_{1,n}, \dots, Y_{z,n}) > 1) + \mathbf{P}(Y_{1,n} = \dots = Y_{z,n} = 0) \\ &\geq 3^{-z}(1 - f_n[1]) .\end{aligned}$$

Now assume $\sum_{n=1}^{\infty} (1 - f_n[1]) = \infty$. Since for fixed z the events $A_{z,n}$ are independent, it follows by the Borel-Cantelli Lemma that these events occur a.s. infinitely often. From the a.s. convergence of Z_n we get for $z \geq 1$

$$\mathbf{P}(Z_{\infty} = z) = \mathbf{P}(Z_n \neq z \text{ finitely often}) \leq \mathbf{P}(A_{z,n} \text{ occurs finitely often}) = 0 .$$

This implies that $\mathbf{P}(1 \leq Z_{\infty} < \infty) = 0$.

Conversely let $\sum_{n=1}^{\infty} (1 - f_n[1]) < \infty$. Then for $z \geq 1$ with $\mathbf{P}(Z_0 = z) > 0$ we have

$$\mathbf{P}(Z_{\infty} = z) \geq \mathbf{P}(Z_n = z \text{ for all } n) \geq \mathbf{P}(Z_0 = z) \left(\prod_{n=1}^{\infty} f_n[1] \right)^z > 0 ,$$

and it follows $\mathbf{P}(1 \leq Z_{\infty} < \infty) > 0$. Thereby the proof is finished. \square

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